Note on: Lot-sizing for inventory systems with product recovery

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Abstract

In this article we revisit the paper by Teunter (2004), appeared in Computers and Industrial Engineering. For this model Teunter proposed an approach leading to an approximate solution. Here we propose an optimization procedure, which leads to policies with integer set up numbers in the production and the remanufacturing facilities i.e. to the optimal policy.

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1. Introduction

Teunter (2004) presented an inventory model, where the stationary demand is satisfied though two modes. One is by new ordered/produced products and the other by recovered used products which recovery brings back to 'as good as new' condition. All relevant costs i.e. ordering/production and recovery set up, holding new/recovered items, holding recoverable items are constant. He considers policies that alternate one production lot with a fixed number *R* of recovery lots respectively, in short (1, *R*) policies and one recovery lot with a fixed number *P* of production lots, in short (*P*, 1) policies. In this class of policies Teunter derived simple closed type formulas for the optimal procurement/production and recovery lot sizes. These formulas are more general than the ones given in Nahmias and Rivera (1979) and Koh, Hwang, Sohn and Ko (2002) as they are valid for infinite and finite recovery and production rates respectively.

The approach used in his analysis is to minimize the total cost function $TC(Q_p, Q_r)$, w.r.t. to the procurement/production Q_p and recovery Q_r lot sizes, treating them as continuous variables. Treating lot sizes as continuous variables, in cases where these have to be integer, is common practice in inventory control literature and really it does not create any problem. He then obtained *R*, *P* using equations connected the above variables. The so obtained values are truncated, if necessary, to the nearest integer and the so obtained policy is applied. In the case of Teunter's model the obtained values of Q_p, Q_r are used to calculate *P* and *R*. If the values for *R* or *P* are not integers the policy cannot be applied. To overcome this difficulty, the author suggests suitable modifications. He first truncates the obtained values of *R*, *P* to the nearest integer, greater or equal to one. Next using these values he modifies the initially obtained values of Q_p in (1, *R*) policies and the Q_r in (*P*, 1) policies. The resulting policy can be applied and the relevant cost is calculated. In this paper we present an approach, which leads directly to the optimal policy with *R*, *P* integers. These values are then used to obtain the lot sizes Q_p and Q_r and to calculate the optimal cost.

2. Model

The notation in Teunter's (2004) model is:

d	Demand rate
f	Return fraction (return rate fd)
p	Production rate
r	Recovery rate
K_p	Ordering (setup) cost per production lot
K_r	Ordering (setup) cost per recovery lot
h_r	Holding cost per recoverable item per time unit
h_s	Holding cost per serviceable item per time unit
Q_p	Production lot size
Q_r	Recovery lot size

2.1. Policy (1,R): One manufacturing against R remanufacturing opportunities

For this class of policies, the total cost per unit of time is given by:

$$TC(Q_p,Q_r)=\frac{K_p d(1-f)}{Q_p}+\frac{K_r df}{Q_r}+\frac{h_s((1-f)(1-d/p)Q_p+f(1-d/r)Q_r)}{2}+\frac{h_r f(Q_p+(1-d/r)Q_r)}{2} \quad (1)$$

The variables Q_r , Q_p and R are connected via the relation

$$RQ_r(1-f)=Q_p f\,. \quad (2)$$

Teunter minimized $TC(Q_p,Q_r)$ w.r.t. Q_p, Q_r and using (2) he obtained *R*. The so obtained *R* is not in general integer. To make it integer, he truncates this *R* to the nearest integer, say $\tilde{R}=\max\{1,[R]\}$, greater or equal to one and using this truncation, he replaces the initially obtained Q_p value by the one obtained though the relation

$$\tilde{Q}_p=\frac{\tilde{R}Q_r(1-f)}{f}.$$

Here we shall approach the solution of this problem in a different way.

From (2) we have that:

$$Q_r = \frac{Q_p f}{R(1-f)}. \tag{3}$$

Replacing this Q_r into (1) the total cost per unit of time becomes:

$$TC(Q_p,R) = \frac{K_p d(1-f) + RK_r d(1-f)}{Q_p} + Q_p\left[\frac{h_s(1-f)(1-d/p)}{2} + \frac{h_r f}{2}\right] + \frac{Q_p f^2(1-d/r)(h_s+h_r)}{2R(1-f)}. \tag{4}$$

If we set

$A = K_p d(1-f) + RK_r d(1-f) = A_1 + A_2 R$, where $A_1 = K_p d(1-f) \geq 0$ and $A_2 = K_r d(1-f) \geq 0$,

$$B = \frac{h_s(1-f)(1-d/p)}{2} + \frac{h_r f}{2} \geq 0 \text{ and } C = \frac{f^2(1-d/r)(h_s+h_r)}{2R(1-f)} = \frac{C_1}{R}, C_1 = \frac{f^2(1-d/r)(h_s+h_r)}{2(1-f)} \geq 0, \tag{5}$$

we can rewrite $TC(Q_p,R)$ as:

$$TC(Q_p,R) = \frac{A}{Q_p} + Q_p(B+C). \tag{6}$$

Now the problem becomes: find the minimum of $TC(Q_p,R)$ w.r.t. R and Q_p. The approach we follow is first finding the minimum of this function w.r.t. Q_p. The minimizing point will be a function of R, say $Q_p(R)$. Replaces this into the objective function and minimize the objective w.r.t. R.

From (6) we see that $TC(Q_p,R)$ is convex in Q_p and so attains its minimum at

$$Q_p^*(R) = \sqrt{\frac{A}{B+C}} = \sqrt{\frac{A_1 + RA_2}{B + \frac{C_1}{R}}}. \tag{7}$$

Substituting $Q_p^*(R)$ into (6) yields:

$$TC(Q_p^*(R), R) = 2\sqrt{A_1 B + A_2 C_1 + A_2 BR + \frac{A_1 C_1}{R}}. \tag{8}$$

Since R is integer we use the difference function

$$\Delta TC(Q_p^*, R) = TC(Q_p^*, R) - TC(Q_p^*, R-1),\ R \geq 2$$

for the location of optimal R which in our case is:

$$\Delta TC(Q_p^*,R)=TC(Q_p^*,R)-TC(Q_p^*,R-1)=\frac{2(A_2B-\frac{A_1C_1}{R(R-1)})}{\sqrt{A_1B+A_2C_1+RA_2B+\frac{A_1C_1}{R}}+\sqrt{A_1B+A_2C_1+(R-1)A_2B+\frac{A_1C_1}{R-1}}} \quad (9)$$

From (9) we see that if $\frac{A_1C_1}{A_2B}\leq 2$, then $\Delta TC(Q_p^*,R)\geq 0$ for any $R\geq 2$ and the optimum is at $R^*=1$.

If this is not the case, then there always exists a $R^*\geq 2$ such that $\Delta TC(Q_p^*,R)<0$ for all $R\leq R^*$ and $\Delta TC(Q_p^*,R)\geq 0$ for all $R>R^*$. Simple algebra on these inequalities gives that this R^* satisfies the double inequality

$$R^*(R^*-1)<\frac{A_1C_1}{A_2B}\leq R^*(R^*+1),\quad R^*\geq 2. \quad (10)$$

In case that $R^*(R^*+1)=\frac{A_1C_1}{A_2B}$, we have two equivalent solutions and we agree to keep the smallest one. The integer value of R^* obtained from (10), is used in (7) to calculate $Q_p^*(R^*)$ and the resulting policy can be implemented to give the cost.

We apply this approach to the example proposed by Teunter. The data of the example are:

d=1000, f=0.8, p=5000, r=3000, K_p=20, K_r=5, h_r=2 and h_s=10.

With these data we have $A_1=4000$, $A_2=1000$, $B=1.6$, $C_1=12.8$ and the cost function is:

$$TC(Q_p,R)=\frac{4000+1000R}{Q_p}+Q_p(1.6+\frac{12.8}{R}).$$

From (10) we get R^*=6 and (7) gives $Q_p^*=51.75$ and finally $TC(Q_p^*,R^*)$=386.44 . From (3) we find $Q_r^*=34.5$. The policy given by Teunter has Q_p^*=53.03, Q_r^*=35.35 and $TC(Q_p^*,R^*)$=386.55. In this example the deviations for the lot sizes and the total cost are negligible. Computational experience shows that the two approaches give quite similar results, in case that the exact R obtained using Teunter's approach is greater than one. In case that this R is smaller than one the deviations are significant. This is evident in the examples given in table 1 and suggests that in this case the approximate approach used by Teunter leads to costs much higher than the optimal.

2.2. Policy (P,1): P manufacturing opportunities against one remanufacturing

The total cost per unit time in this case is:

$$TC(Q_p,Q_r)=\frac{K_p d(1-f)}{Q_p}+\frac{K_r df}{Q_r}+\frac{h_s((1-f)(1-d/p)Q_p+f(1-d/r)Q_r)}{2}+\frac{h_r(1-fd/r)Q_r}{2} \quad (11)$$

and P is fully determined by the lot-sizes via the relation

$$Q_r(1-f)=PQ_pf\,. \quad (12)$$

Teunter minimized $TC(Q_p,Q_r)$ w.r.t. Q_p,Q_r and using (12) he obtained P. The so obtained P is not in general integer. To make it integer, he truncates this P to the nearest integer, say $\tilde{P}=\max\{1,[P]\}$, greater or equal to one and using this truncation, he replaces the initially obtained Q_r value by the one obtained though the relation:

$$\tilde{Q}_r=\frac{\tilde{P}Q_pf}{1-f}.$$

Here we shall approach the solution of this problem in a different way.

From (12) we have that:

$$Q_p=\frac{Q_r(1-f)}{Pf}. \quad (13)$$

So using (13), the total cost per unit (11) becomes:

$$TC(Q_r,P)=\frac{PK_pdf+K_rdf}{Q_r}+Q_r[\frac{h_s(1-f)^2(1-d/p)}{2Pf}+\frac{h_sf(1-d/r)}{2}+\frac{h_r(1-fd/r)}{2}]. \quad (14)$$

If we set

$A=PK_pdf+K_rdf=PA_1+A_2$, where $A_1=K_pdf\geq 0$ and $A_2=K_rdf\geq 0$,

$B=\frac{h_s(1-f)^2(1-d/p)}{2Pf}+\frac{h_sf(1-d/r)}{2}+\frac{h_r(1-fd/r)}{2}=\frac{B_1}{P}+B_2$,

where $B_1=\frac{h_s(1-f)^2(1-d/p)}{2f}\geq 0$ and $B_2=\frac{h_sf(1-d/r)}{2}+\frac{h_r(1-fd/r)}{2}\geq 0$ (15)

we can rewrite $TC(Q_r,P)$ as:

$$TC(Q_r, P) = \frac{A}{Q_r} + Q_r B. \tag{16}$$

From (16), we see that $TC(Q_r, P)$ is convex in Q_r and attains its minimum at

$$Q_r^*(P) = \sqrt{\frac{A}{B}} = \sqrt{\frac{PA_1 + A_2}{\frac{B_1}{P} + B_2}}. \tag{17}$$

Substituting (17) into (16) yields:

$$TC(Q_r^*(P), P) = 2\sqrt{A_1B_1 + A_2B_2 + A_1B_2P + \frac{A_2B_1}{P}}. \tag{18}$$

The difference function

$$\Delta TC(Q_r^*, P) = TC(Q_r^*, P) - TC(Q_r^*, P-1), \; P \geq 2$$

of $TC(Q_r^*, R)$ shows that:

$$\Delta TC(Q_r^*, P) = TC(Q_r^*, P) - TC(Q_r^*, P-1) = \frac{2(A_1B_2 - \frac{A_2B_1}{P(P-1)})}{\sqrt{A_1B_1 + A_2B_2 + PA_1B_2 + \frac{A_2B_1}{P}} + \sqrt{A_1B_1 + A_2B_2 + (P-1)A_1B_2 + \frac{A_2B_1}{P-1}}}. \tag{19}$$

Following the same reasoning as previously we can see that if

$$0 < \frac{A_2B_1}{A_1B_2} \leq 2$$

then the optimum is at $P^* = 1$. If this is not the case, then there always exists a $P^* \geq 2$ such that $\Delta TC(Q_r^*, P) < 0$ for all $P \leq P^*$ and $\Delta TC(Q_r^*, P) \geq 0$ for all $P > P^*$. Simple algebra on these inequalities gives that this P^* satisfies the double inequality

$$P^*(P^* - 1) < \frac{A_2B_1}{A_1B_2} \leq P^*(P^* + 1), \; P^* \geq 2. \tag{20}$$

In case that $P^*(P^* + 1) = \frac{A_2B_1}{A_1B_2}$, we have two equivalent solutions and we agree to always take the smallest one.

For the Teunter's example we have that: $A_1 = 16000$, $A_2 = 4000$, $B_1 = 0.2$ and $B_2 = 3.4$. The cost function is:

$$TC(Q_r,P)=\frac{16000P+4000}{Q_r}+Q_r(3.4+\frac{0.2}{P}).$$

From (20), we get $P^*=1$ and (17) gives $Q_r^*=74.54$ and finally $TC(Q_r^*,P^*)$=536.66. From (13) we find that $Q_p^*=18.63$. The policy given by Teunter has Q_p^*=70.71, Q_r^*=282.8 and $TC(Q_p^*,R^*)$=1088.9. In this example the deviations for the lot sizes and the total cost are very significant. Computational experience shows that the two approaches give quite similar results, in case that the exact *P* obtained using Teunter's approach is greater than one. In case that this *P* is smaller than one the deviations are significant. This is evident in the examples given in table 2 and suggests that in this case the approximate approach used by Teunter leads to costs much higher than the optimal.

3. Conclusion

In this paper we propose a solution method for Teunter's (2004) model which leads to integer values for the parameters *R, P* in the set of policies (1, *R)* and (*P*,1) and subsequently to the optimal policy. This is an exact approach and comparing the results obtained, to those given by Teunter's approximate method, we see that in some cases Teunter's algorithm performs very well, while in other cases the cost deviations from the optimal are significant and his method should not be applied.

References

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Data								Teunter's exact solution				Teunter's approximated solution				Proposed optimal solution				% Cost Deviation
d	f	p	r	K_p	K_r	h_r	h_s	Q_p	Q_r	R	$TC(Q_p,Q_r)$	R	Q_p	Q_r	$TC(Q_p,Q_r)$	R	Q_p	Q_r	$TC(Q_p,Q_r)$	Penalty[a]
1000	0,8	5000	3000	20	5	2	10	50,00	35,36	5,66	386,27	6	53,03	35,36	386,55	6	51,75	34,50	386,44	0,0003
1000	0,2	4000	2500	10	5	4	6	60,30	40,83	0,37	314,32	1	163,29	40,83	457,24	1	71,46	17,86	335,86	0,2655
500	0,3	1000	700	10	10	5	8	40,35	51,89	0,33	231,31	1	121,07	51,89	347,03	1	54,13	23,20	258,62	0,2548
500	0,3	2000	1000	20	12	10	12	38,80	33,03	0,50	469,83	1	77,06	33,03	558,19	1	45,72	19,59	489,90	0,1223
800	0,5	2400	1500	20	8	4	15	47,81	37,99	1,26	503,11	1	37,99	37,99	511,98	1	44,26	44,26	506,07	0,0115
1000	0,7	3000	2000	20	20	2	10	59,41	81,65	1,69	544,92	2	69,98	81,65	547,64	2	65,86	76,83	546,63	0,0018
20	0,8	50	35	30	20	5	6	7,13	13,03	2,19	82,79	2	6,51	13,03	82,93	2	6,76	13,51	82,87	0,0007
20	0,8	100	50	30	25	2	7	9,39	13,61	2,76	84,34	3	10,20	13,61	84,43	3	9,95	13,27	84,40	0,0004
20	0,2	80	60	50	30	4	20	11,18	8,66	0,32	170,82	1	34,64	8,66	272,51	1	13,72	3,43	186,59	0,3152

[a]penalty=(Teunter's approximated $TC(Q_p,Q_r)$ - proposed optimal $TC(Q_p,Q_r)$)/ Teunter's approximated $TC(Q_p,Q_r)$

Table 1. Policy (1, *R*)

Data								Teunter's exact solution				Teunter's approximated solution				Proposed optimal solution				% Cost Deviation
d	f	p	r	K_p	K_r	h_r	h_s	Q_p	Q_r	P	$TC(Q_p,Q_r)$	P	Q_p	Q_r	$TC(Q_p,Q_r)$	P	Q_p	Q_r	$TC(Q_p,Q_r)$	Penalty[a]
1000	0,8	5000	3000	20	5	2	10	70,71	34,30	0,12	346,38	1	70,71	282,8	1088,94	1	18,63	74,54	536,66	0,5072
1000	0,2	4000	2500	10	5	4	6	66,67	21,32	1,28	333,81	1	66,67	16,67	336,67	1	71,46	17,87	335,86	0,0024
500	0,3	1000	700	10	10	5	8	50,00	25,50	1,19	257,66	1	50,00	21,43	259,44	1	54,13	23,20	258,62	0,0032
500	0,3	2000	1000	20	12	10	12	47,14	18,70	0,93	489,55	1	47,14	20,20	490,13	1	45,72	19,60	489,90	0,0005
800	0,5	2400	1500	20	8	4	15	56,57	31,54	0,56	485,76	1	56,57	56,57	521,37	1	44,26	44,26	506,07	0,0293
1000	0,7	3000	2000	20	20	2	10	77,46	76,38	0,42	521,53	1	77,46	180,7	666,15	1	42,64	99,49	562,85	0,1551
20	0,8	50	35	30	20	5	6	18,26	11,58	0,16	68,41	1	18,26	73,03	191,76	1	4,49	17,98	89,01	0,5358
20	0,8	100	50	30	25	2	7	14,64	13,02	0,22	161,41	1	14,64	58,55	161,41	1	4,69	18,76	93,81	0,4188
20	0,2	80	60	50	30	4	20	11,50	6,12	2,12	177,82	2	11,55	5,77	177,82	2	11,70	5,85	177,81	0,0000

[a]penalty=(Teunter's approximated $TC(Q_p,Q_r)$ - proposed optimal $TC(Q_p,Q_r)$)/ Teunter's approximated $TC(Q_p,Q_r)$

Table 2. Policy (*P*, 1)